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LETTER TO THE EDITOR

R-matrices and generalized inverses

H W Braden†

Department of Mathematics and Statistics, The University of Edinburgh, Edinburgh, UK

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Abstract. Four results are given that address the existence, ambiguities and construction of a classical R-matrix given a Lax pair. They enable the uniform construction of R-matrices in terms of any generalized inverse of $\mathrm{ad}\, L$. For generic L a generalized inverse (and indeed the Moore–Penrose inverse) is explicitly constructed. The R-matrices are, in general, momentum dependent and dynamical. The construction applies equally to Lax matrices with spectral parameter.

1. Introduction

The modern approach to completely integrable systems is in terms of Lax pairs L, M and R-matrices. Here the consistency of the matrix equation $\dot{L} = [L, M]$ expresses the equations of motion of the system under consideration. The great merit of this approach is that it provides a unified framework for treating the many disparate completely integrable systems known. Given a $2n$-dimensional phase space, Liouville's theorem [1, 12], which ensures the existence of action-angle variables, requires that we have n independent conserved quantities in involution; that is, they mutually Poisson commute. As a consequence of the Lax equation the traces $\mathrm{Tr}\, L^k$ are conserved and these are natural candidates for the action variables of Liouville's theorem. (In practice the action variables are typically transcendental functions of these traces.) It remains, however, to verify that these traces provide enough independent quantities in involution. Verifying the number of independent quantities is usually straightforward and the remaining step is then to show they mutually Poisson commute. The final ingredient of the modern approach, the R-matrix [15], guarantees their involution. If L is in a representation E of a Lie algebra $\mathfrak{g}$ (here taken to be semi-simple), the classical R-matrix is a $E \otimes E$ valued matrix such that

$$[R, L \otimes 1] - [R^{\mathrm{T}}, 1 \otimes L] = \{L \overset{\otimes}{,} L\}. \tag{1}$$

(The notation is amplified below.) Then

$$\{\mathrm{Tr}_E\, L^k, \mathrm{Tr}_E\, L^m\} = \mathrm{Tr}_{E\otimes E}\{L^k \overset{\otimes}{,} L^m\} = \mathrm{km}\, \mathrm{Tr}_{E\otimes E}\, L^{k-1} \otimes L^{m-1}\{L \overset{\otimes}{,} L\} = 0$$

which vanishes due to the cyclicity of the trace. By a result of Babelon and Viallet [4], such an R-matrix is guaranteed to exist if the eigenvalues of L are in involution. The Liouville integrability of a system represented by a Lax pair has been reduced then to finding any solution to (1) and counting the number of independent traces. Further, the R-matrix is an essential ingredient when examining the separation of variables of such integrable systems [11, 16].

† E-mail address: hwb@ed.ac.uk

Unfortunately the construction of R-matrices has hitherto been somewhat of an arcane art and many have been obtained in a case by case manner [3]. The purpose of this letter is to present four results that address the existence and construction of solutions of (1) and hence the Liouville integrability of the system under consideration. They yield a uniform construction of R-matrices. In fact the R-matrices satisfying (1) are by no means unique and our construction characterizes this ambiguity. The approach applies equally to R-matrices with spectral parameter. We will illustrate these results with a simple example. At the outset we remark that the R-matrix solutions to (1) are generically momentum dependent. Within this family of solutions some may be particularly simple: they may, for example, be constant (as in the Toda system [9]) or momentum independent. We are content here with providing the construction of an R-matrix given a Lax matrix L and so answering the question of Liouville integrability: we do not seek to further specify the momentum or position dependence of the solution. For the elliptic Calogero–Moser model there are in fact [6] no R-matrices that are independent of both momentum and spectral parameter (for more than four particles) and this illustrates the fact that simple assumptions on the parameter dependence of an R-matrix need not be natural. Elsewhere we will apply these results to the elliptic Calogero–Moser models without spectral parameter.

Our approach is as follows. First, we rewrite (1) in the form of the matrix equation

$$A^{\mathrm{T}}X - X^{\mathrm{T}}A = B. \tag{2}$$

Here A is built out of L and the Lie algebra, the unknown matrix X being solved for is essentially the R-matrix in a given basis and B represents the right-hand side of (1). Our first result is to give necessary and sufficient conditions for (2) to admit solutions together with its general solution. This general solution encodes the possible ambiguities of the R-matrix. Because A is (in general) singular our solution is in terms of a generalized inverse G satisfying

$$AGA = A \qquad \text{and} \qquad GAG = G. \tag{3}$$

Such a generalized inverse always exists. (Accounts of generalized inverses may be found in [5, 8, 13, 14].) Indeed the Moore–Penrose inverse—which is unique and always exists—further satisfies $(AG)^{\dagger} = AG$, $(GA)^{\dagger} = GA$. Observe that given a G satisfying (3) we have at hand projection operators $P_1 = GA$ and $P_2 = AG$ which satisfy

$$AP_1 = P_2A = A \qquad P_1G = GP_2 = G. \tag{4}$$

Our second result shows that the choice of generalized inverse G only alters the R-matrix within the ambiguities specified by the general solution, and so any generalized inverse suffices to solve (2) and hence construct an R-matrix. At this stage we have reduced the problem of constructing an R-matrix to that of constructing a generalized inverse G and our third result constructs such for a generic element L of $\mathfrak{g}$. Because the Moore–Penrose inverse is unique, our fourth result is to present this inverse for generic L though we shall not need to use this in our application.

This letter is organized as follows. In the next section we present the four results. The proofs of the first two are somewhat lengthy and algebraic and will be presented elsewhere [7]; the proofs of the remaining two are easier to outline. In section 3 we apply these to give the R-matrix for generic L. In section 4 we extend the results to include a spectral parameter. Section 5 is an illustrative example. We conclude with a brief discussion.

2. Four results

Our first task is to identify (1) with (2). Let T_μ denote a basis for the (finite-dimensional) Lie algebra $\mathfrak{g}$ with $[T_\mu, T_\nu] = c^\lambda_{\mu\nu}\, T_\lambda$ defining the structure constants of $\mathfrak{g}$. Set $\phi(T_\mu) = X_\mu$, where ϕ yields the representation E of the Lie algebra $\mathfrak{g}$; we may take this to be a faithful representation. With $L = \sum_\mu L^\mu X_\mu$ the left-hand side of (1) becomes

$$\{L \overset{\otimes}{,} L\} = \sum_{\mu,\nu} \{L^\mu, L^\nu\} X_\mu \otimes X_\nu$$

while upon setting $R = R^{\mu\nu} X_\mu \otimes X_\nu$ and $R^{\mathrm{T}} = R^{\nu\mu} X_\mu \otimes X_\nu$ the right-hand side yields

$$\begin{aligned}[R, L \otimes 1] - [R^{\mathrm{T}}, 1 \otimes L] &= R^{\mu\nu}([X_\mu, L] \otimes X_\nu - X_\nu \otimes [X_\mu, L]) \\ &= R^{\mu\nu} L^\lambda([X_\mu, X_\lambda] \otimes X_\nu - X_\nu \otimes [X_\mu, X_\lambda]) \\ &= (R^{\tau\nu} c^\mu_{\tau\lambda} L^\lambda - R^{\tau\mu} c^\nu_{\tau\lambda} L^\lambda) X_\mu \otimes X_\nu .\end{aligned}$$

By identifying $A^{\mu\nu} = c^\nu_{\mu\lambda} L^\lambda \equiv -\mathrm{ad}\, L^\nu_\mu$, $B^{\mu\nu} = \{L^\mu, L^\nu\}$ and $X^{\mu\nu} = R^{\mu\nu}$ we see that (1) is an example of (2).

Having shown how to identify (1) with the matrix equation (2) we may now state our first result.

Result 1. The matrix equation (2) has solutions if and only if

(C1) $\quad B^{\mathrm{T}} = -B$

(C2) $\quad (1 - P_1^{\mathrm{T}}) B (1 - P_1) = 0$

in which case the general solution is

$$X = \tfrac{1}{2} G^{\mathrm{T}} B P_1 + G^{\mathrm{T}} B (1 - P_1) + (1 - P_2^{\mathrm{T}}) Y + (P_2^{\mathrm{T}} Z P_2) A \tag{5}$$

where Y is arbitrary and Z is only constrained by the requirement that $P_2^{\mathrm{T}} Z P_2$ be symmetric.

Although the general solution appears to depend on the generalized inverse G we in fact find:

Result 2. If $\bar{G}$ is any other solution of (3) with attendant projection operators $\bar{P}_{1,2}$ then (5) may also be written

$$X = \tfrac{1}{2} \bar{G}^{\mathrm{T}} B \bar{P}_1 + \bar{G}^{\mathrm{T}} B (1 - \bar{P}_1) + (1 - \bar{P}_2^{\mathrm{T}}) \bar{Y} + \bar{P}_2^{\mathrm{T}} \bar{Z} \bar{P}_2 A$$

where

$$\bar{Y} = (1 - P_2^{\mathrm{T}}) Y + P_2^{\mathrm{T}} Z P_2 A + G^{\mathrm{T}} B (1 - \tfrac{1}{2} P_1) \qquad \bar{Z} = Z + \tfrac{1}{2}(G^{\mathrm{T}} B \bar{G} - \bar{G}^{\mathrm{T}} B G).$$

Thus $\bar{Z}$ is again symmetric and we have a solution of the form (5).

In the R-matrix context the matrix B is manifestly antisymmetric because of the antisymmetry of the Poisson bracket and so (C1) is clearly satisfied. We have thus reduced the existence of an R-matrix to the single consistency equation (C2) and the construction of a generalized inverse to $\mathrm{ad}\, L$. We turn now to the construction of the generalized inverse.

Let X_μ denote a Cartan-Weyl basis for the Lie algebra $\mathfrak{g}$. That is $\{X_\mu\} = \{H_i, E_\alpha\}$, where $\{H_i\}$ is a basis for the Cartan subalgebra $\mathfrak{h}$ and $\{E_\alpha\}$ is the set of step operators (labelled by the root system Φ of $\mathfrak{g}$). The structure constants are found from

$$\begin{aligned}&[H_i, E_\alpha] = \alpha_i E_\alpha \qquad [E_\alpha, E_{-\alpha}] = \alpha^\vee \cdot H \qquad \text{and} \\ &[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \qquad \text{if } \alpha + \beta \in \Phi .\end{aligned}$$

Here $N_{\alpha,\beta} = c^{\alpha+\beta}_{\alpha\,\beta}$. With these definitions we then have that

$$\mathrm{ad}\, L = \begin{array}{c c} & \begin{array}{cc} \overset{j}{\downarrow} & \overset{\beta}{\downarrow} \end{array} \\ \begin{array}{c} i \rightarrow \\ \alpha \rightarrow \end{array} & \begin{pmatrix} 0 & -\beta_i^{\vee} L^{-\beta} \\ -\alpha_j L^{\alpha} & \Lambda^{\alpha}_{\beta} \end{pmatrix} \end{array} = \begin{pmatrix} 0 & u^{\mathrm{T}} \\ v & \Lambda \end{pmatrix} \tag{6}$$

where we index the rows and columns first by the Cartan subalgebra basis $\{i, j : 1 \ldots \mathrm{rank}\, \mathfrak{g}\}$ then the root system $\{\alpha, \beta \in \Phi\}$. We will use this block decomposition of matrices throughout. Here u and v are $|\Phi| \times \mathrm{rank}\, \mathfrak{g}$ matrices and we have introduced the $|\Phi| \times |\Phi|$ matrix

$$\Lambda^{\alpha}_{\beta} = \alpha \cdot L\, \delta^{\alpha}_{\beta} + c^{\alpha}_{\alpha-\beta\beta}\, L^{\alpha-\beta} \tag{7}$$

where $\alpha \cdot L = \sum_{i=1}^{\mathrm{rank}\, \mathfrak{g}} \alpha_i L^i$. With these definitions we have:

Result 3. For generic L the matrix Λ is invertible and a generalized inverse of $\mathrm{ad}\, L$ is given by

$$\begin{pmatrix} 1 & 0 \\ -\Lambda^{-1} v & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & -u^{\mathrm{T}}\Lambda^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}. \tag{8}$$

We establish the result by first showing that for generic L

$$\mathrm{ad}\, L = \begin{pmatrix} 1 & u^{\mathrm{T}}\Lambda^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Lambda^{-1} v & 1 \end{pmatrix}. \tag{9}$$

It then follows that (8) is a generalized inverse for $\mathrm{ad}\, L$ by direct multiplication.

Now for any matrices m and Λ we have the general factorization [5, 10]

$$\begin{pmatrix} m & u^{\mathrm{T}} \\ v & \Lambda \end{pmatrix} = \begin{pmatrix} 1 & u^{\mathrm{T}}\Xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m - u^{\mathrm{T}}\Xi v & u^{\mathrm{T}}(1 - \Xi\Lambda) \\ (1 - \Lambda\Xi) v & \Lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Xi v & 1 \end{pmatrix}$$

where Ξ is a generalized inverse of Λ. In particular, when $m = 0$ and Λ is invertible (and so $\Xi = \Lambda^{-1}$) this shows that

$$\begin{pmatrix} 0 & u^{\mathrm{T}} \\ v & \Lambda \end{pmatrix} = \begin{pmatrix} 1 & u^{\mathrm{T}}\Lambda^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -u^{\mathrm{T}}\Lambda^{-1} v & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Lambda^{-1} v & 1 \end{pmatrix}. \tag{10}$$

Thus (9) and hence the result follow by establishing that Λ is generically invertible and that

$$u^{\mathrm{T}}\Lambda^{-1} v = 0. \tag{11}$$

From (7) we see that Λ is the perturbation of a diagonal matrix and so is generically invertible: the zero locus $\det \Lambda = 0$ is a polynomial in the coefficients of $\mathrm{ad}\, L$ and so the complement of this set is dense and open. For such an invertible Λ we thus have

$$\mathrm{rank}\, \Lambda = \dim \Lambda = \dim \mathfrak{g} - \mathrm{rank}\, \mathfrak{g}. \tag{12}$$

Now the maximum rank† of the matrix $\mathrm{ad}\, L$ is $\dim \mathfrak{g} - \mathrm{rank}\, \mathfrak{g}$ [17]. From (10) we see that $\mathrm{rank}\, \Lambda + \mathrm{rank}(u^{\mathrm{T}}\Lambda^{-1} v) = \mathrm{rank}\, \mathrm{ad}\, L \leqslant \dim \mathfrak{g} - \mathrm{rank}\, \mathfrak{g}$ and so from (12) we deduce that $\mathrm{rank}\, \Lambda = \mathrm{rank}\, \mathrm{ad}\, L$. Therefore, $\mathrm{rank}(u^{\mathrm{T}}\Lambda^{-1} v) = 0$ and consequently (11) must hold. The result then follows.

An alternate factorization of $\mathrm{ad}\, L$ is possible for the generic L under consideration. Utilizing (11) we find that

$$\mathrm{ad}\, L = \begin{pmatrix} u^{\mathrm{T}}\Lambda^{-1} \\ 1 \end{pmatrix} \Lambda \begin{pmatrix} \Lambda^{-1} v & 1 \end{pmatrix} = E \Lambda F.$$

Employing a result of MacDuffee (see [5]) this full rank factorization then yields:

† If $\det(t - \mathrm{ad}\, L) = \sum_{j=0}^{\dim \mathfrak{g}} p_j(L)\, t^j$ is the characteristic polynomial of $\mathrm{ad}\, L$, the regular semi-simple elements of a semi-simple Lie algebra $\mathfrak{g}$ are those elements for which $p_{\mathrm{rank}\, \mathfrak{g}}(L) \neq 0$. These elements are also of rank $\dim \mathfrak{g} - \mathrm{rank}\, \mathfrak{g}$ and form an open dense set in $\mathfrak{g}$, but this condition is different from $\det \Lambda \neq 0$.

Result 4. For generic L the Moore–Penrose inverse of $\mathrm{ad}\,L$ is given by

$$F^{\dagger}(F\,F^{\dagger})^{-1}\Lambda^{-1}(E^{\dagger}\,E)^{-1}E^{\dagger}$$

where

$$E = \begin{pmatrix} u^{\mathrm{T}}\Lambda^{-1} \\ 1 \end{pmatrix} \qquad \text{and} \qquad F = (\,\Lambda^{-1}v \quad 1\,)\,.$$

3. The R-matrix

We now bring together the results of the previous section to present the R-matrix for a generic L when this exists. From the fact that $A = -(\mathrm{ad}L)^{\mathrm{T}}$ a generalized inverse of A is given by minus the transpose of the generalized inverse (8). Utilizing our earlier notation this means that we have the projectors

$$P_1 = \begin{pmatrix} 0 & 0 \\ \Lambda^{-1\mathrm{T}}u & 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & v^{\mathrm{T}}\Lambda^{-1\mathrm{T}} \\ 0 & 1 \end{pmatrix}.$$

Let us express the Poisson brackets of the entries of L in the same block form in the Cartan–Weyl basis:

$$B = \begin{pmatrix} \zeta & -\mu^{\mathrm{T}} \\ \mu & \phi \end{pmatrix} = -B^{\mathrm{T}}$$

where $B^{\alpha j} = \{L^{\alpha}, L^{j}\} = \mu_{\alpha j}$ and so on. The constraint (C1) is manifestly satisfied.

The constraint (C2) is now (the rank $\mathfrak{g} \times$ rank $\mathfrak{g}$ matrix equation)

$$\text{(C2)} \qquad 0 = \zeta + \mu^{\mathrm{T}}\Lambda^{-1\mathrm{T}}u - u^{\mathrm{T}}\Lambda^{-1}\mu + u^{\mathrm{T}}\Lambda^{-1}\phi\,\Lambda^{-1\mathrm{T}}u. \tag{13}$$

Each term in this equation is known and so the equality may be readily checked.

Supposing the constraint (C2) is satisfied we then find from (5) that the general R-matrix takes the form

$$R = \begin{pmatrix} 0 & 0 \\ -\Lambda^{-1}\mu + \frac{1}{2}\Lambda^{-1}\phi\Lambda^{-1\mathrm{T}}u & -\frac{1}{2}\Lambda^{-1}\phi \end{pmatrix} + \begin{pmatrix} p & q \\ -\Lambda^{-1}vp - Fu & -\Lambda^{-1}vq - F\Lambda^{\mathrm{T}} \end{pmatrix}. \tag{14}$$

The second term characterizes the ambiguity in R where we have parametrized the matrices Y and Z in (5) by

$$Y = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \qquad \text{and} \qquad Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here the matrices p and q are arbitrary while the entries of Z are such that

$$F = \Lambda^{-1}vav^{\mathrm{T}}\Lambda^{-1\mathrm{T}} + d + \Lambda^{-1}vb + cv^{\mathrm{T}}\Lambda^{-1\mathrm{T}} \tag{15}$$

is symmetric.

4. Inclusion of spectral parameter

For simplicity we have presented our construction for Lax pairs with no spectral parameter but it is straightforward to incorporate such a parameter. The relevant equation to be solved for is now

$$\{L(u) \overset{\otimes}{,} L(v)\} = [R(u,v), L(u)\otimes 1] - [R^{\pi}(u,v), 1\otimes L(v)] \tag{16}$$

where if $R(u,v) = R^{\mu\nu}(u,v)X_\mu \otimes X_\nu$ then $R^\pi(u,v)$ is defined by† $R^\pi(u,v) = R^{\nu\mu}(v,u)X_\mu \otimes X_\nu$. Now

$$B^{\mu\nu}(u,v) = \{L^\mu(u), L^\nu(v)\} = -B^{\nu\mu}(v,u)$$

and because $L(u)$ depends on u alone the generalized inverse now also depends on the spectral parameter as $G = G(u)$. The equation we now wish to solve is

$$A^{\mathrm{T}}(u)X(u,v) - X^\pi(u,v)A(v) = A^{\mathrm{T}}(u)X(u,v) - X^{\mathrm{T}}(v,u)A(v) = B(u,v)$$

and this has the analogous solution

$$X(u,v) = \tfrac{1}{2}G^{\mathrm{T}}(u)B(u,v)P_1(v) + G^{\mathrm{T}}(u)B(u,v)(1-P_1(v)) + (1-P_2^{\mathrm{T}}(u))Y(u,v) + (P_2^{\mathrm{T}}(u)Z(u,v)P_2(v))A(v)$$

if and only if

$$\text{(C1}'\text{)} \qquad B^\pi = -B$$
$$\text{(C2}'\text{)} \qquad (1-P_1^{\mathrm{T}}(u))B(u,v)(1-P_1(v)) = 0.$$

Here $Y(u,v)$ is arbitrary while the symmetry condition now becomes

$$(P_2^{\mathrm{T}}(u)Z(u,v)P_2(v))^\pi = P_2^{\mathrm{T}}(u)Z(u,v)P_2(v).$$

As in the spectral parameter independent case, this reduces to the requirement that

$$F(u,v) = \Lambda^{-1}(u)v(u)a(u,v)v^{\mathrm{T}}(v)\Lambda^{-1\mathrm{T}}(v) + d(u,v) + \Lambda^{-1}(u)v(u)b(u,v) + c(u,v)v^{\mathrm{T}}(v)\Lambda^{-1\mathrm{T}}(v) \tag{17}$$

be such that $F^\pi = F$.

5. An example

We conclude with the simple but illustrative example of the harmonic oscillator presented as the Lax pair (with spectral parameter)

$$L(u) = \begin{pmatrix} \mathrm{i}px/u & (p^2/u)+1 \\ (x^2/u)+1 & -\mathrm{i}px/u \end{pmatrix} \qquad M(u) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{18}$$

The consistency of the Lax equation $\dot{L}(u) = [L(u), M(u)]$ follows from the equations of motion of the Hamiltonian $H = (p^2 + x^2 + u)/2 = -(u/2)\det L(u)$.

Although we could equally work with the simple algebra $su(2)$ in this example we will take the algebra to be $gl(2)$. Now for any $l \in gl(2)$,

$$l = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = aH_1 + bE_{12} + cE_{21} + dH_2$$

we find that in our Cartan–Weyl basis

$$\mathrm{ad}\, l = \begin{pmatrix} 0 & 0 & -c & b \\ 0 & 0 & c & -b \\ -b & b & a-d & 0 \\ c & -c & 0 & d-a \end{pmatrix}.$$

Then

$$u = \begin{pmatrix} -c & c \\ b & -b \end{pmatrix} \qquad v = \begin{pmatrix} -b & b \\ c & -c \end{pmatrix} \qquad \Lambda = \begin{pmatrix} a-d & 0 \\ 0 & d-a \end{pmatrix}$$

† We use π to denote both matrix transposition together with the interchange of u and v while T denotes ordinary matrix transposition.

and l is generic† provided $a-d \neq 0$. We note that in the Cartan–Weyl basis the permutation operator $P = \sum_i H_i \otimes H_i + \sum_{\alpha\in\Phi} E_\alpha \otimes E_{-\alpha}$ (which is such that $P(X\otimes Y)P = Y\otimes X$) takes the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{19}$$

Now for the case at hand $b = (p^2/u) + 1$, $c = (x^2/u) + 1$, $a = -d = \mathrm{i}px/u$ and L is generic for $px \neq 0$. We may, therefore, use the expressions computed in sections 3 and 4. We calculate (using $\{p, x\} = 1$) that

$$B(u, v) = \frac{1}{uv}\begin{pmatrix} 0 & 0 & -2\mathrm{i}p^2 & 2\mathrm{i}x^2 \\ 0 & 0 & 2\mathrm{i}p^2 & -2\mathrm{i}x^2 \\ 2\mathrm{i}p^2 & -2\mathrm{i}p^2 & 0 & 4xp \\ -2\mathrm{i}x^2 & 2\mathrm{i}x^2 & -4xp & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mu^{\mathrm{T}}(u, v) \\ \mu(u, v) & \phi(u, v) \end{pmatrix}.$$

and straightforwardly verify that condition (C2′) is satisfied. The R-matrix is then given by

$$R(u, v) = \begin{pmatrix} 0 & 0 \\ -\Lambda^{-1}(u)\mu(u, v) + \frac{1}{2}\Lambda^{-1}(u)\phi(u, v)\Lambda^{-1\mathrm{T}}(u)u(v) & -\frac{1}{2}\Lambda^{-1}(u)\phi(u, v) \end{pmatrix}$$
$$+ \begin{pmatrix} p(u, v) & q(u, v) \\ -\Lambda^{-1}(u)v(u)p(u, v) - F(u, v)u(v) & -\Lambda^{-1}(u)v(u)q(u, v) - F(u, v)\Lambda^{\mathrm{T}}(v) \end{pmatrix}.$$

The second term again characterizes the ambiguity in R and we have parametrized the matrices $Y(u, v)$ and $Z(u, v)$ in an analogous way to the spectral parameter independent case of section 3. Substitution of the various quantities gives for the first term

$$R(u, v) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{-p^2+v}{2pxv} & -\dfrac{-p^2+v}{2pxv} & 0 & \dfrac{\mathrm{i}}{v} \\ \dfrac{-x^2+v}{2pxv} & -\dfrac{-x^2+v}{2pxv} & \dfrac{\mathrm{i}}{v} & 0 \end{pmatrix}.$$

This R-matrix is clearly dynamical. Making use of the the block structure of the R-matrix we see that by choosing

$$p(u, v) = \frac{-2\mathrm{i}}{u-v}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad q(u, v) = 0 \quad \text{and} \quad F(u, v) = -\frac{u+v}{u-v}\frac{1}{2px}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we arrive at the non-dynamical

$$R(u, v) = \frac{-2\mathrm{i}}{u-v}P$$

where P is given by (19).

6. Discussion

We have presented a uniform construction for a classical R-matrix given a Lax pair, thus answering the question of the Liouville integrability of the system in terms of the invariants of the matrix L. The method not only gives necessary and sufficient conditions for the R-matrix to exist and describes its ambiguities, but is algorithmic as well. Given L, first

† It is regular semi-simple provided $(a-d)^2 + bc \neq 0$.

construct ad L. Next construct any generalized inverse to ad L and verify (C2); this is the necessary and sufficient condition for an R-matrix to exist: it is given explicitly by (5). Furthermore, we have given a generalized inverse for generic L in (8); genericity is easily checked by evaluating det $\Lambda \neq 0$, where Λ is the restriction of ad L to the root space (given by (7)). The ambiguities in the R-matrix have been specified. We remark, that the block nature of the R-matrix allows us to easily verify the putative ansatz for a given R-matrix.

Thus far our discussion has been limited to linear R-matrices and we briefly discuss the application to quadratic r-matrices, i.e. the solutions to

$$\{L \overset{\otimes}{,} L\} = [r, L \otimes L] = [r_A, L \otimes L] \tag{20}$$

where $r_A = (r - r^{\mathrm{T}})/2$. (It follows from the antisymmetry of $\{,\}$ that $[r + r^{\mathrm{T}}, L \otimes L] = 0$.) As discussed in [4], the quadratic R-matrix calculation may be reduced to the linear R-matrix situation. In particular the R-matrix

$$R = \tfrac{1}{2} r_A^{\mu\nu} L^{\lambda} (X_\mu \otimes X_\nu X_\lambda + X_\mu \otimes X_\lambda X_\nu) \tag{21}$$

that satisfies (1) yields a solution r_A of (20); the general solution is then built from r_A and the centralizer of $L \otimes L$. Our theorem has given us the left-hand side of (21) and a quadratic r-matrix is then given by solving the linear equation $R^{\mu\sigma} = r_A^{\mu\nu} F_\nu^\sigma$ where $F_\nu^\sigma = (F_{\nu\lambda}^\sigma + F_{\lambda\nu}^\sigma) L^\lambda / 2$ and $X_\nu X_\lambda = F_{\nu\lambda}^\sigma X_\sigma$. Whereas the linear R-matrix involves only Lie algebraic data, the quadratic r-matrix may involve the group structure through the multiplication $X_\nu X_\lambda = F_{\nu\lambda}^\sigma X_\sigma$. Nonetheless, the quadratic r-matrix has been reduced to a linear equation amenable to direct solution.

Finally, we mention that for systems obtained by Hamiltonian reduction an alternative geometric construction of classical R-matrices exists [2] in terms of Dirac brackets. This suggests there is a correspondence between Dirac brackets and generalized inverses. This is indeed the case and I will present this elsewhere.

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